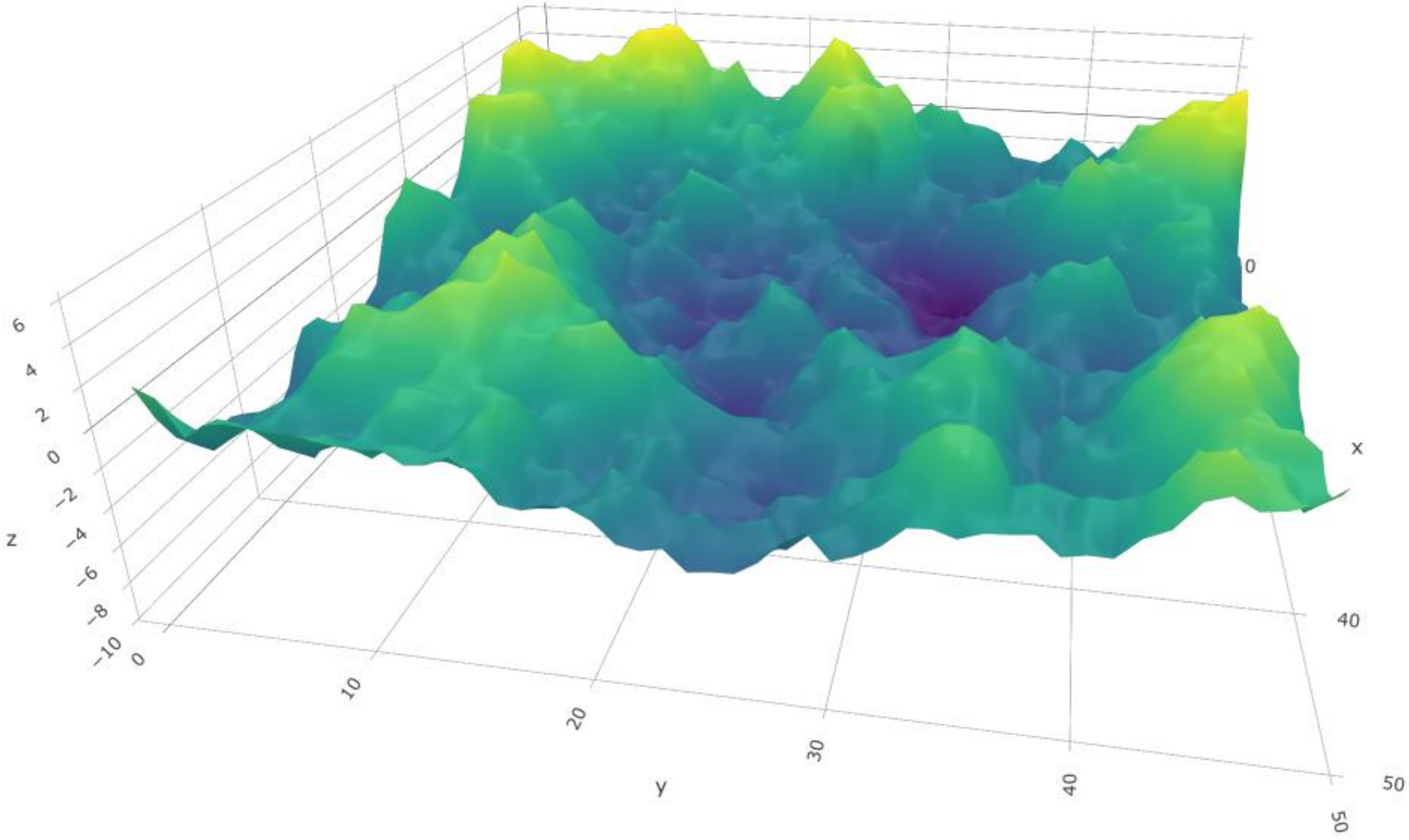
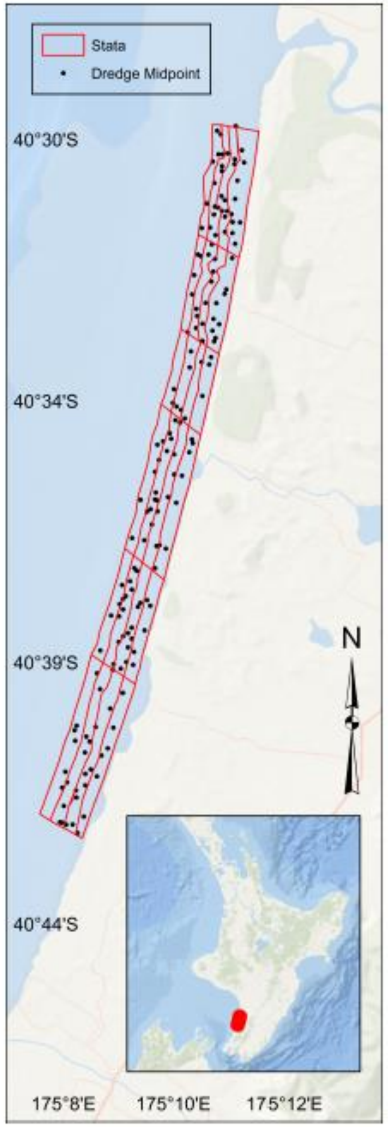
Spatial Modeling with Template Model Builder

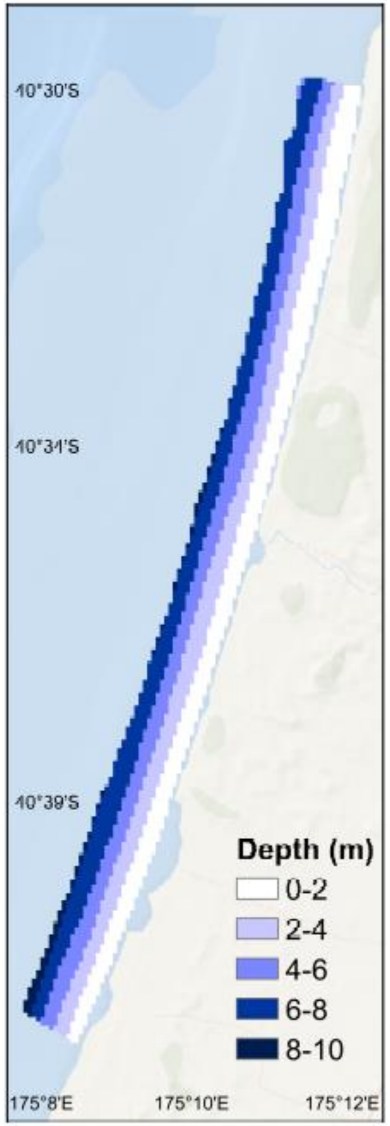
Theory and Mechanics



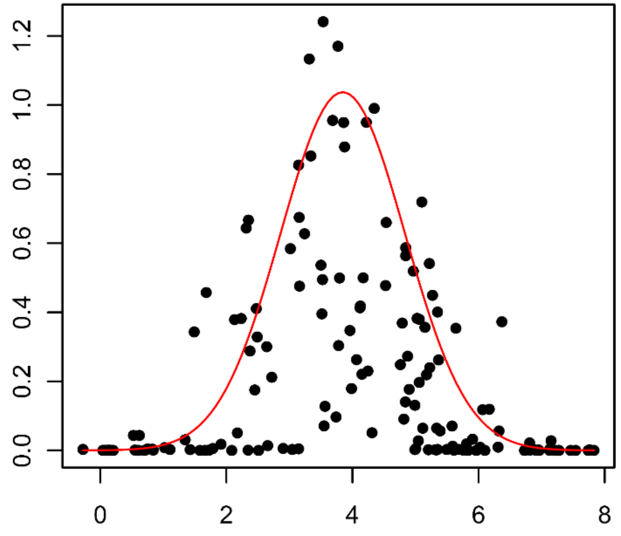
Andrea Havron, OST

Species Distribution Modeling

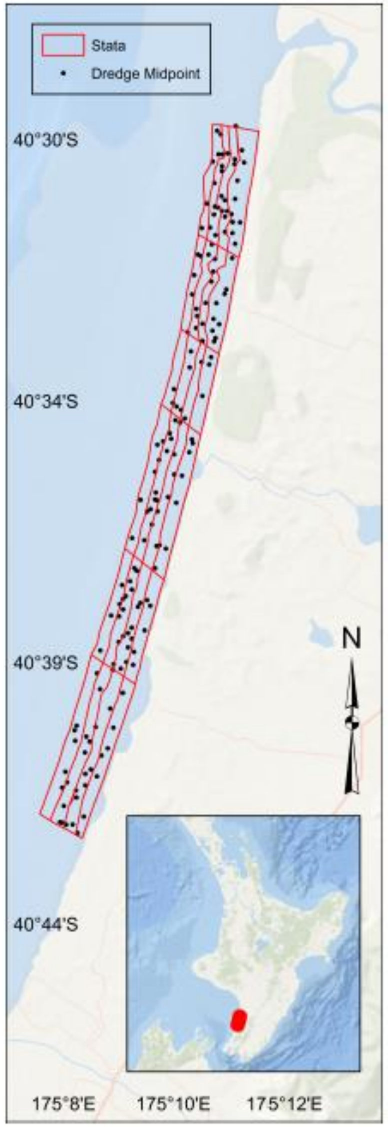


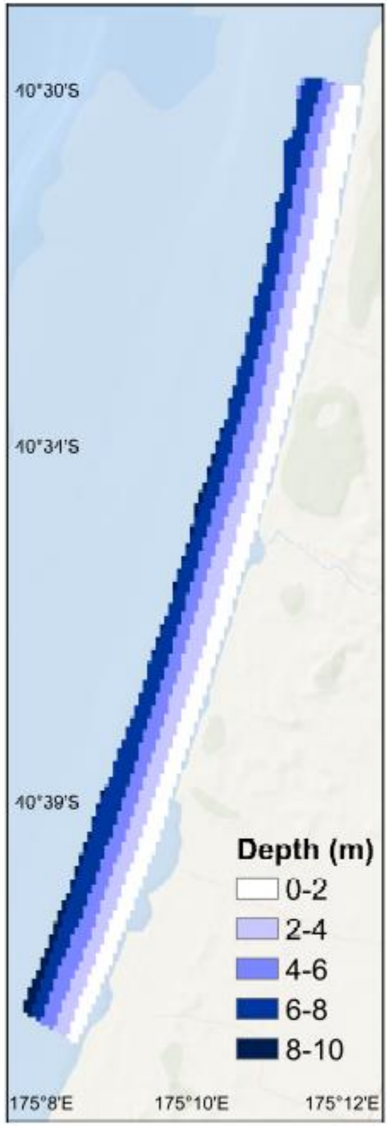


Triangle Shell Surf clam
Manawatu, NZ

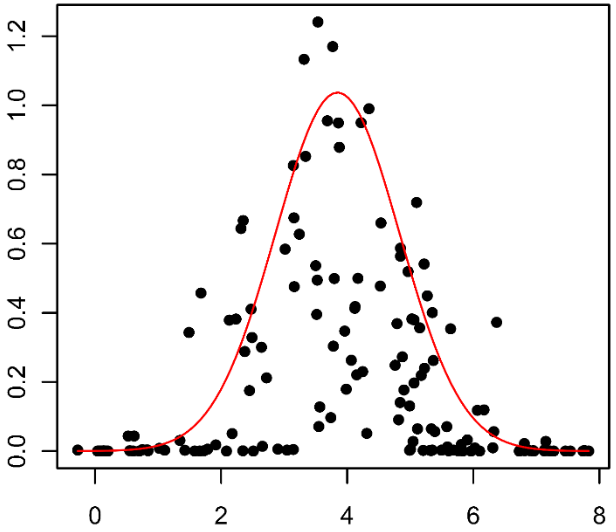


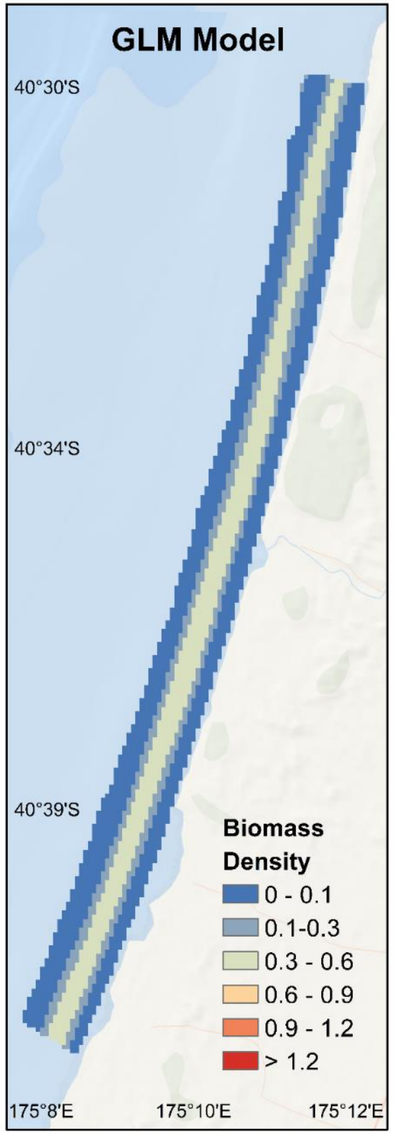
Species Distribution Modeling

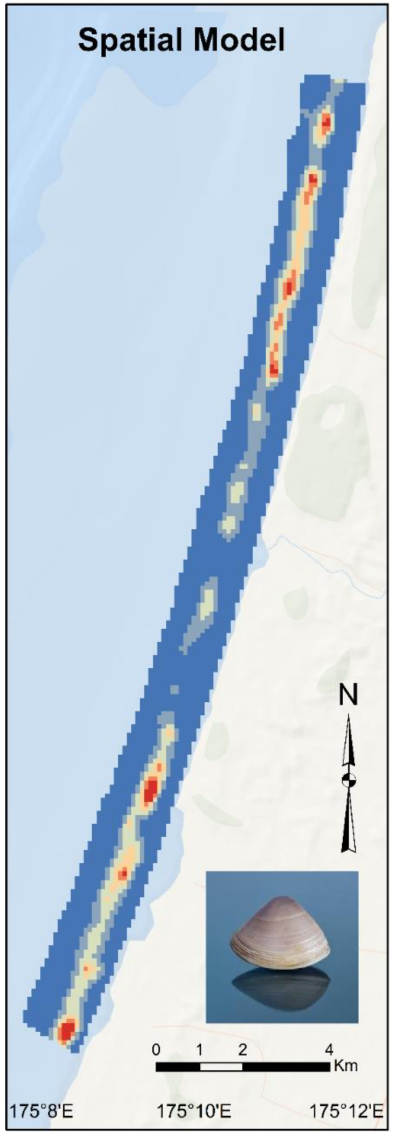




Triangle Shell Surf clam
Manawatu, NZ







Benefits of modeling spatial correlation

- Failure to model spatial correlation violates independence assumptions
  - Narrower confidence intervals
  - Inflated Type I error
- We want to learn about the spatial structure
- More realistic predictions of species distributions

Hierarchical Spatial Model

$$\mathrm{Y} \sim f_y\left(g^{-1}(\mu)\right)$$

$$\mu = \beta_0 + X\beta + \omega$$

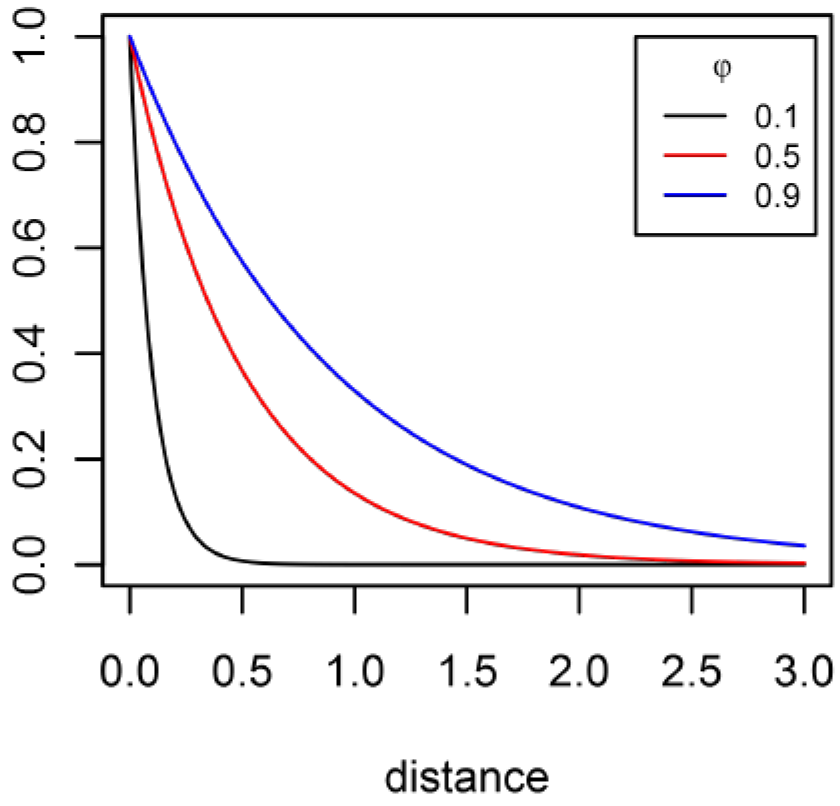
$$\omega \sim MVN(0, \Sigma)$$

$$L(\boldsymbol{\omega}) = \frac{|\Sigma|^{-1/2}}{\sqrt{(2\pi)^n}} \exp(-\frac{1}{2}\boldsymbol{\omega}^T\Sigma^{-1}\boldsymbol{\omega})$$

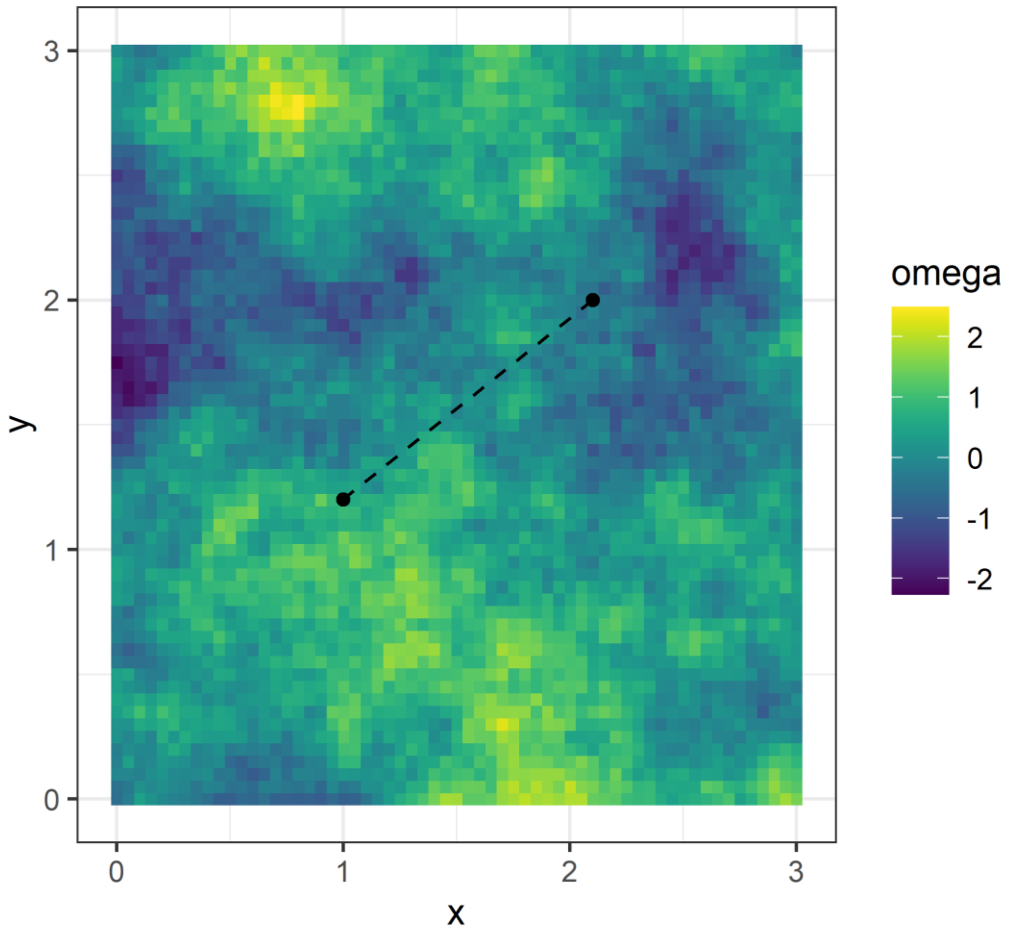
Covariance Matrix, Σ

Exponential Correlation

$$\Sigma_{i,j} = \exp(-\frac{d_{ij}}{\phi})$$



ϕ: distance spatial correlation ≈10%

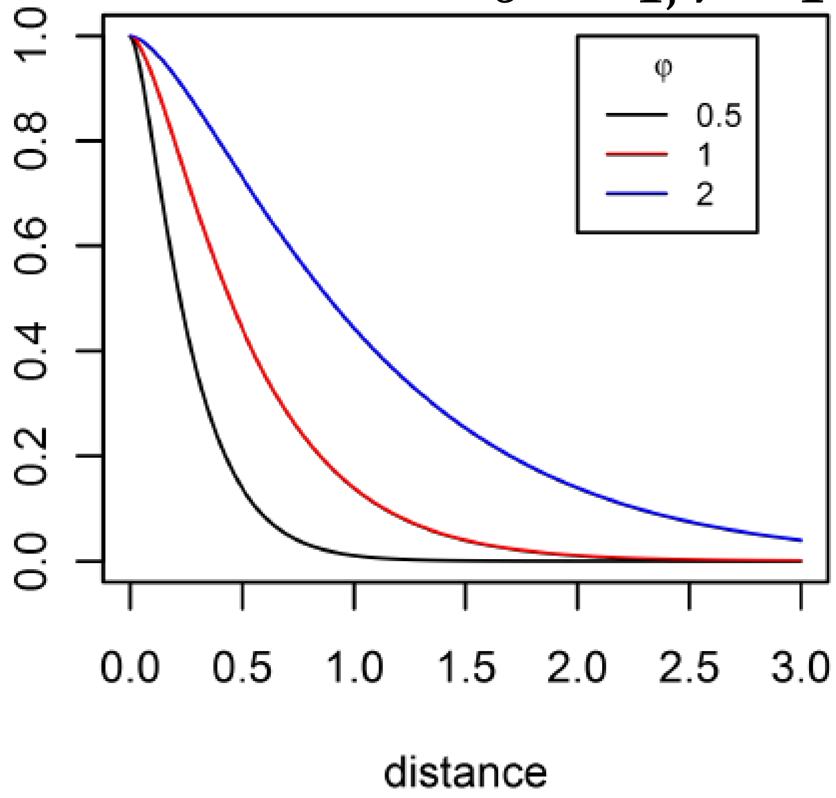


Covariance Matrix, Σ

Matérn Correlation

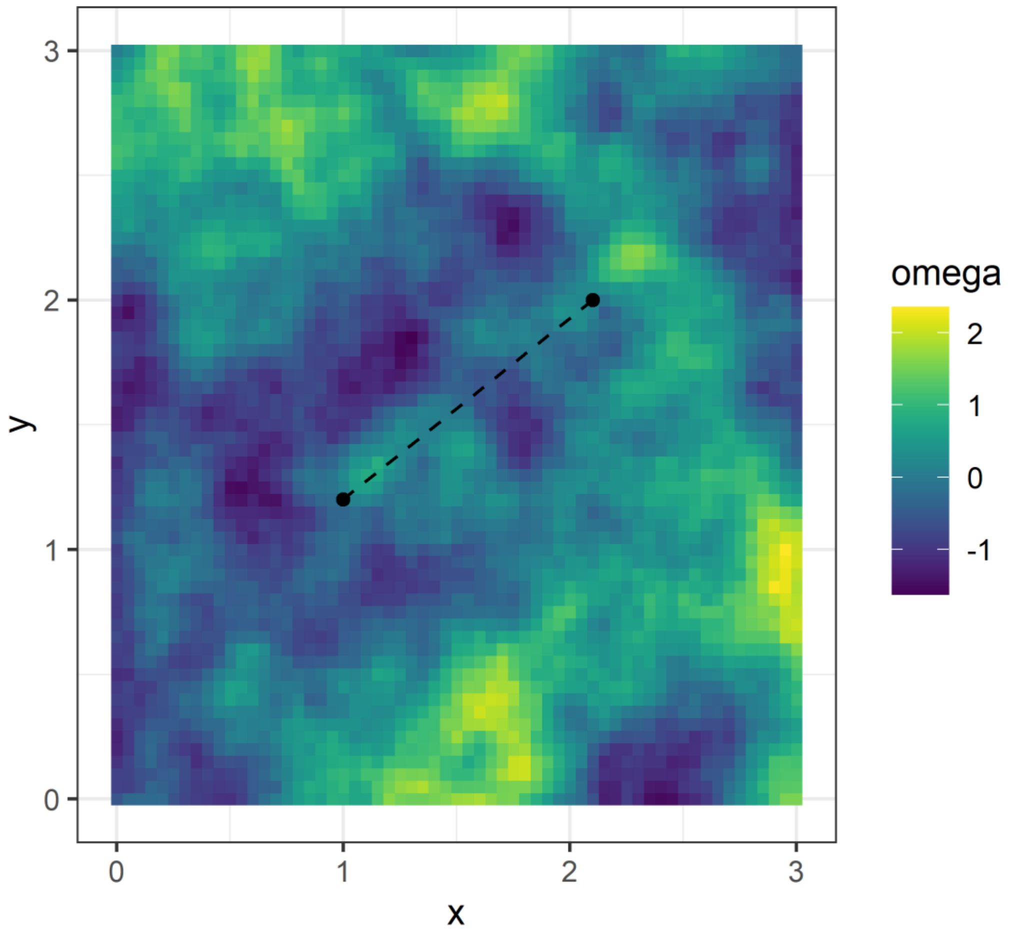
$$\Sigma_{i,j} = \frac{1}{2^{\nu-1}\Gamma(\nu)}(\kappa d)^{\nu} K_{\nu}(\kappa d)$$

$\sigma^2 = 1, \nu = 1$

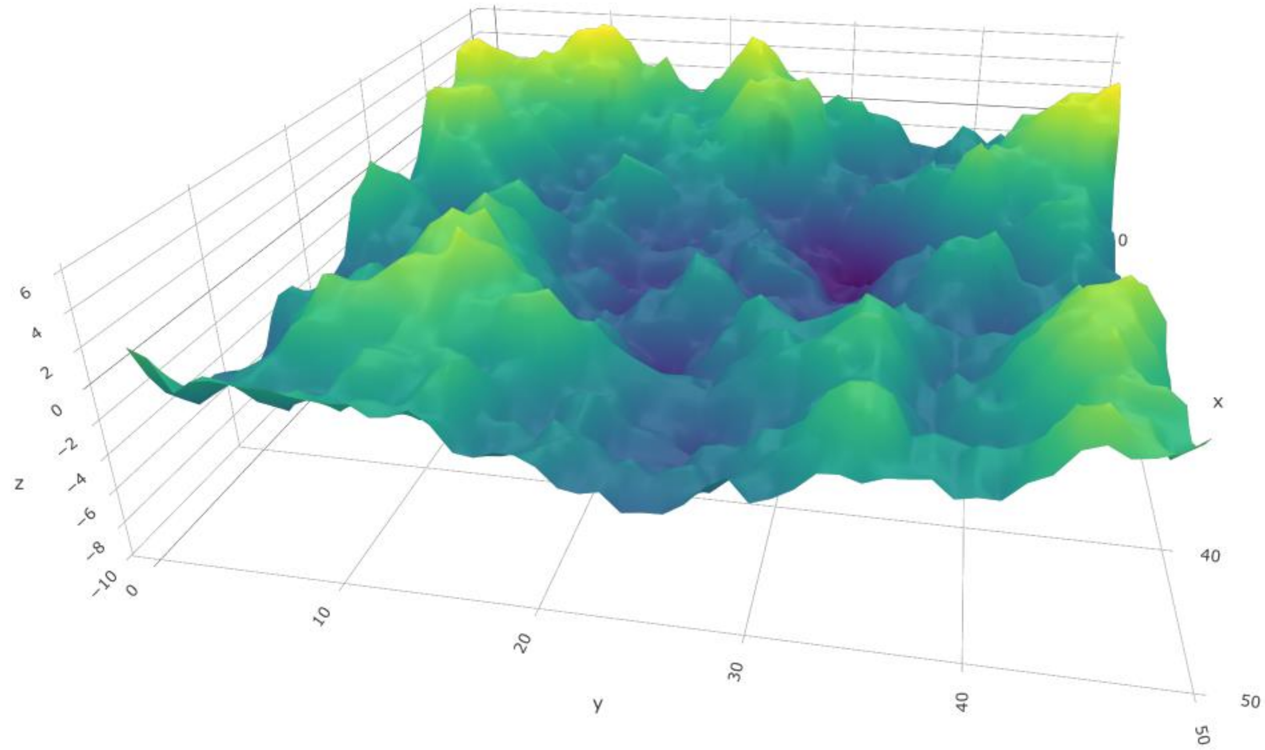


ϕ: distance spatial correlation ≈10%

$\kappa = \sqrt{8\nu}/\phi$



Gaussian Field



$$L(\boldsymbol{\omega}) = \frac{|\Sigma|^{-1/2}}{\sqrt{(2\pi)^n}} \exp(-\frac{1}{2}\boldsymbol{\omega}^T\Sigma^{-1}\boldsymbol{\omega})$$

Likelihood Bottleneck

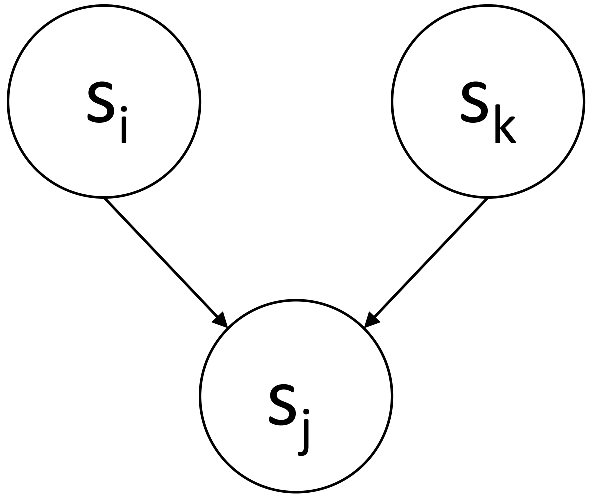
$$L(\boldsymbol{\omega}) = \frac{|\Sigma|^{-1/2}}{\sqrt{(2\pi)^n}} \exp(-\frac{1}{2}\boldsymbol{\omega}^T\Sigma^{-1}\boldsymbol{\omega})$$

- Operations are $\mathcal{O}(n^3)$

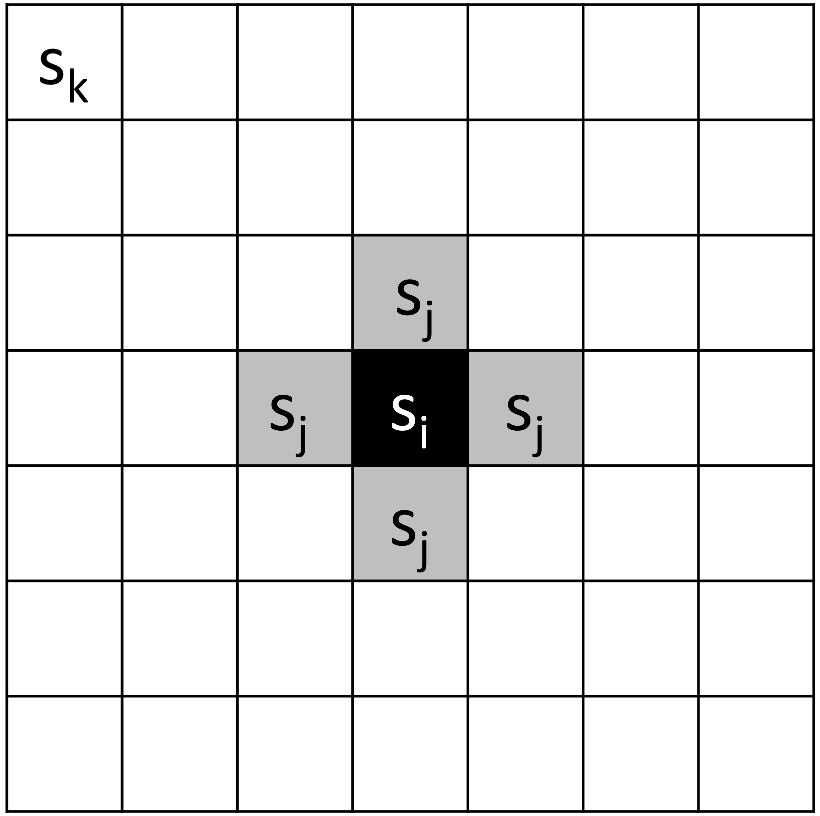
Gaussian Markov Random Field (Besag, 1974)

GMRF (Besag, 1974)

$$Y(s_i|\boldsymbol{s}_{-i}) = Y(s_i|\boldsymbol{s}_j : j \in Nb_i)$$

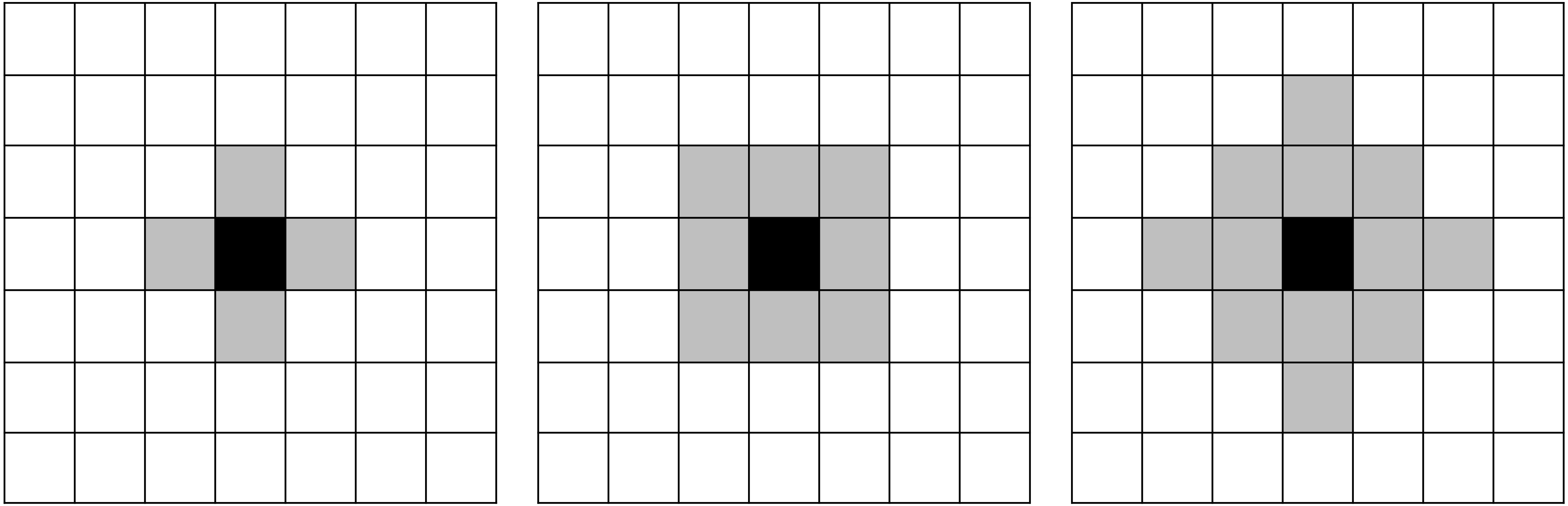


$$s_i \perp s_k \mid s_j$$



GMRF

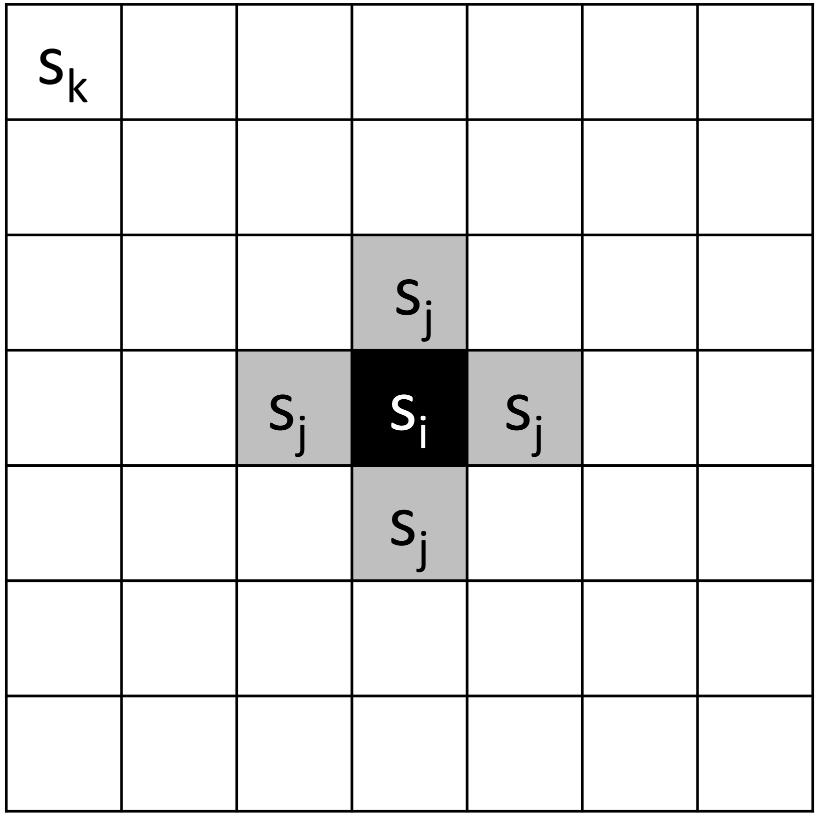
AR(1) AR(2) AR(3)



GMRF

$$Q = \frac{1}{\sigma^2}\begin{bmatrix} 1+\rho^2 & -\rho & \cdot & \cdot \\ -\rho & 1+\rho^2 & -\rho & \cdot \\ \cdot & -\rho & 1+\rho^2 & -\rho \\ \cdot & \cdot & -\rho & 1+\rho^2 \end{bmatrix}$$

AR(1), 1D

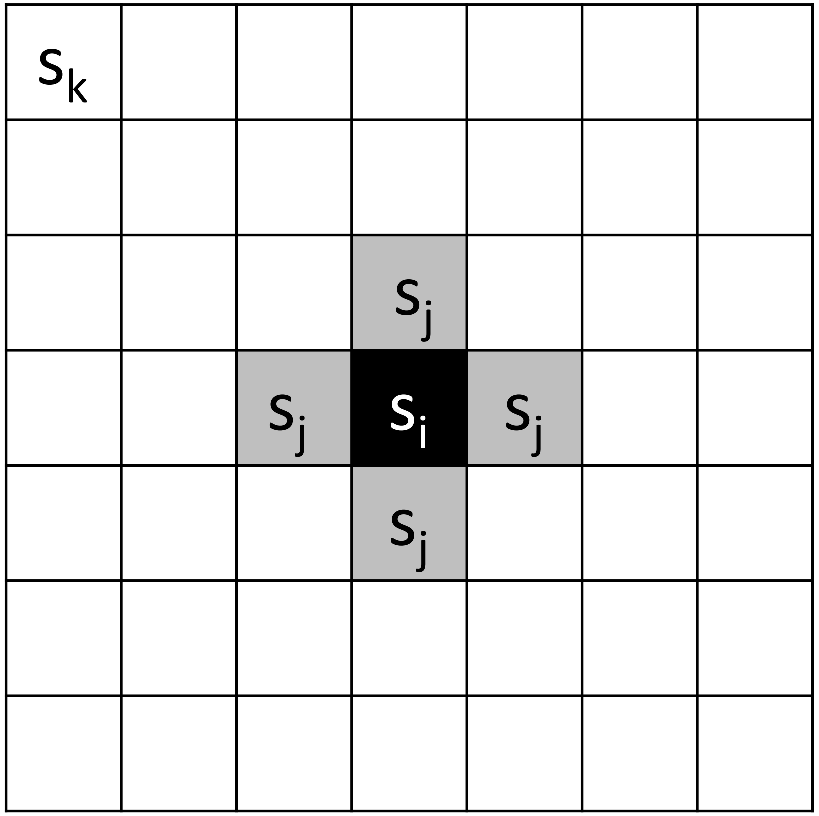


GMRF

$$Q = \frac{1}{\sigma^2}\begin{bmatrix} 1+\rho^2 & -\rho & \cdot & \cdot \\ -\rho & 1+\rho^2 & -\rho & \cdot \\ \cdot & -\rho & 1+\rho^2 & -\rho \\ \cdot & \cdot & -\rho & 1+\rho^2 \end{bmatrix}$$

$$Q = \Sigma^{-1}$$

AR(1), 1D



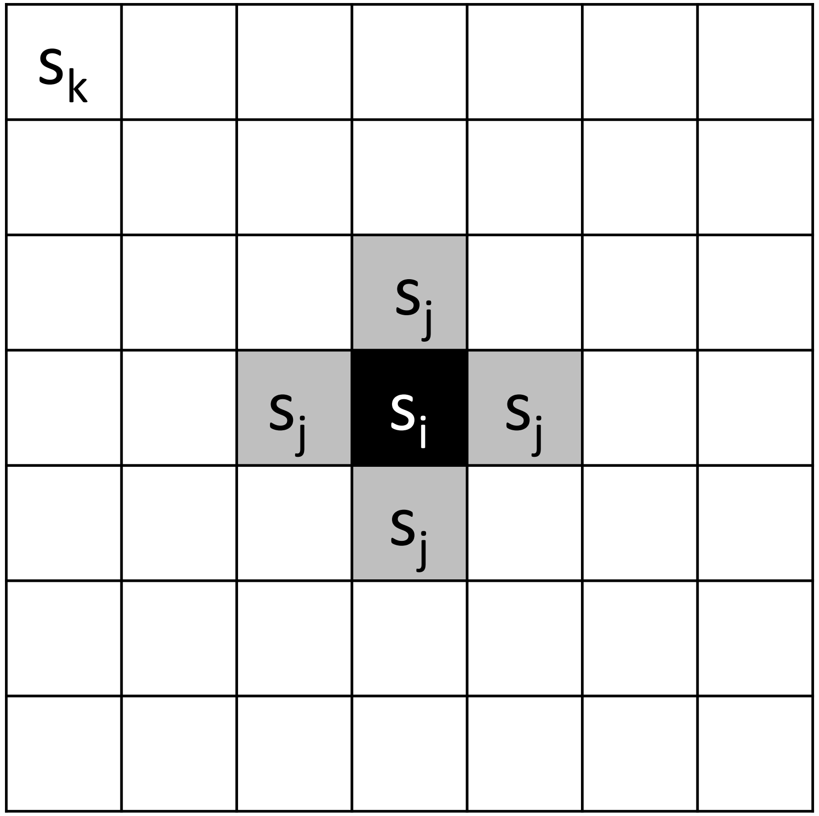
GMRF

$$Q = \frac{1}{\sigma^2}\begin{bmatrix} 1+\rho^2 & -\rho & \cdot & \cdot \\ -\rho & 1+\rho^2 & -\rho & \cdot \\ \cdot & -\rho & 1+\rho^2 & -\rho \\ \cdot & \cdot & -\rho & 1+\rho^2 \end{bmatrix}$$

$$Q = \Sigma^{-1}$$

$$\Sigma = \frac{\sigma^2}{1-\rho^2}\begin{bmatrix} 1 & \rho & \rho^2 & \rho^3 \\ \rho & 1 & \rho & \rho^2 \\ \rho^2 & \rho & 1 & \rho \\ \rho^3 & \rho^2 & \rho & 1 \end{bmatrix}$$

AR(1), 1D



Likelihood Bottleneck

$$L(\boldsymbol{\omega}) = \frac{|\Sigma|^{-1/2}}{\sqrt{(2\pi)^n}} \exp(-\frac{1}{2}\boldsymbol{\omega}^T \boldsymbol{\Sigma}^{-\mathbf{1}} \boldsymbol{\omega})$$

- Operations are $\mathcal{O}(n^3)$

Resolving the Likelihood Bottleneck

$$L(\boldsymbol{\omega}) = \frac{|\Sigma|^{-1/2}}{\sqrt{(2\pi)^n}} \exp(-\frac{1}{2}\boldsymbol{\omega}^T \boldsymbol{Q} \boldsymbol{\omega})$$

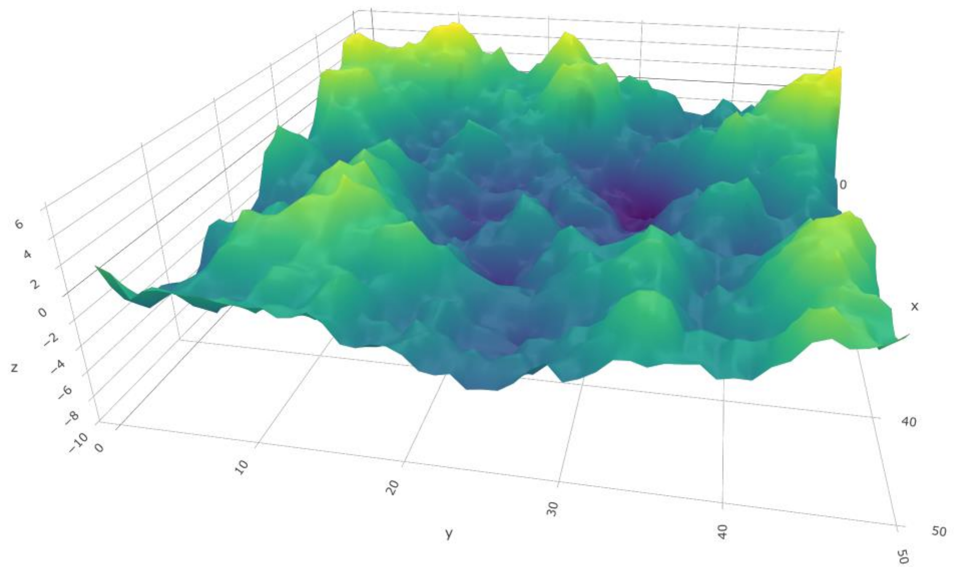
- Operations are $\mathcal{O}(n^{3/2})$

Stochastic Partial Differential Equation

SPDE (Whittle, 1963)

$$(\kappa^2 - \Delta)^{\alpha/2} x(s) = W(s)$$

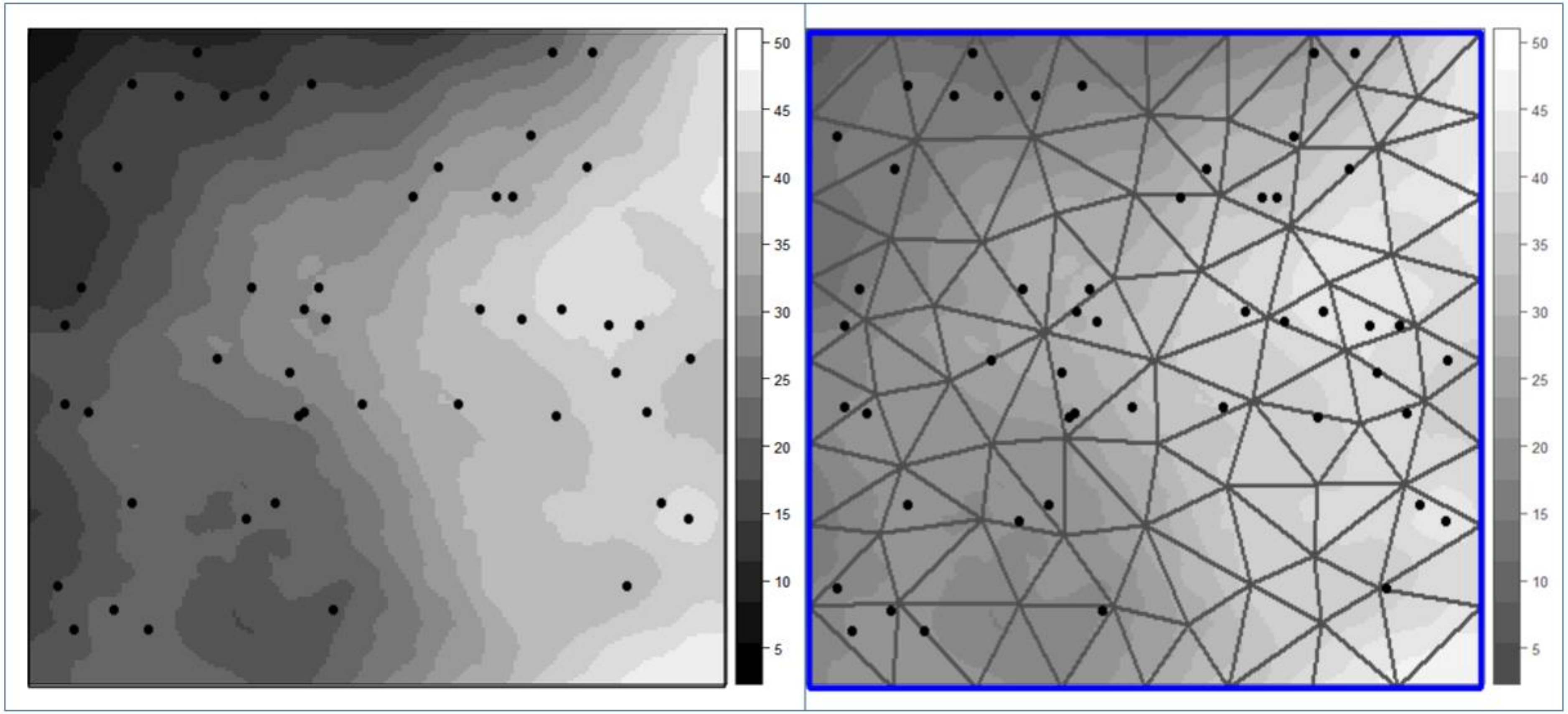
Solution: Gaussian Field with a Matérn covariance matrix



Finite Element Method approach to the SPDE

FEM-SPDE (Lindgren et al., 2011)

Triangulated Mesh



Finite Element Method approach to the SPDE

FEM-SPDE (Lindgren et al., 2011)

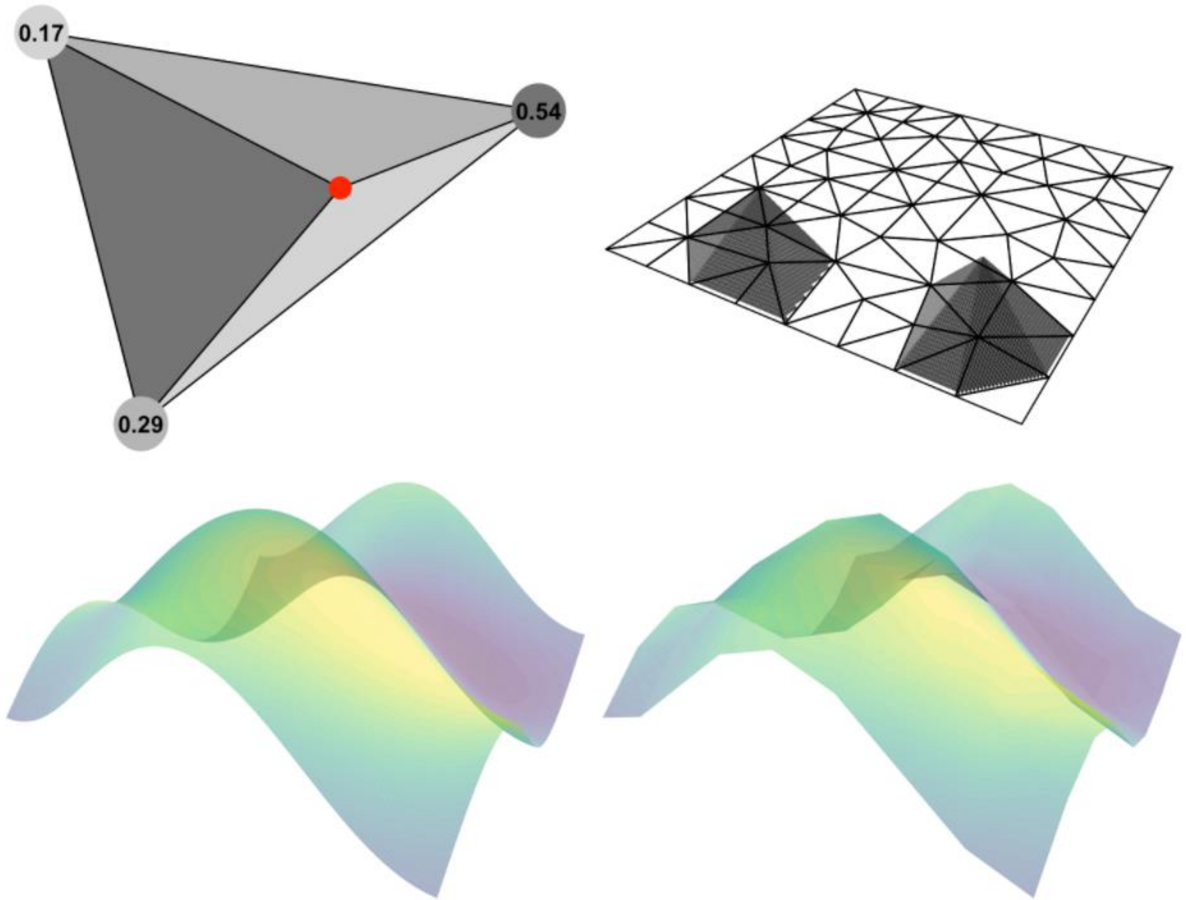
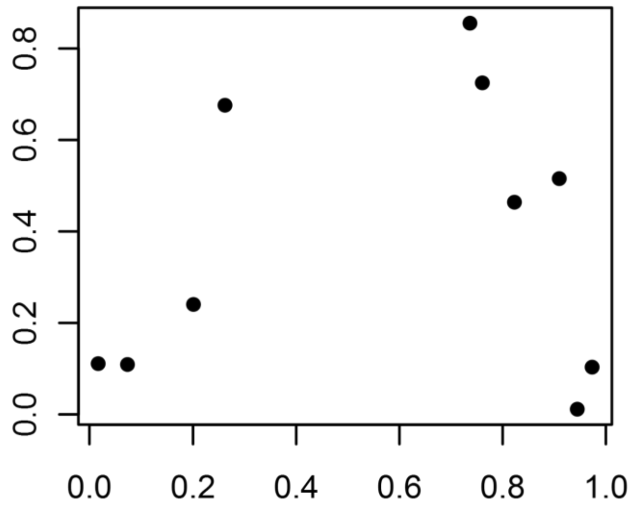
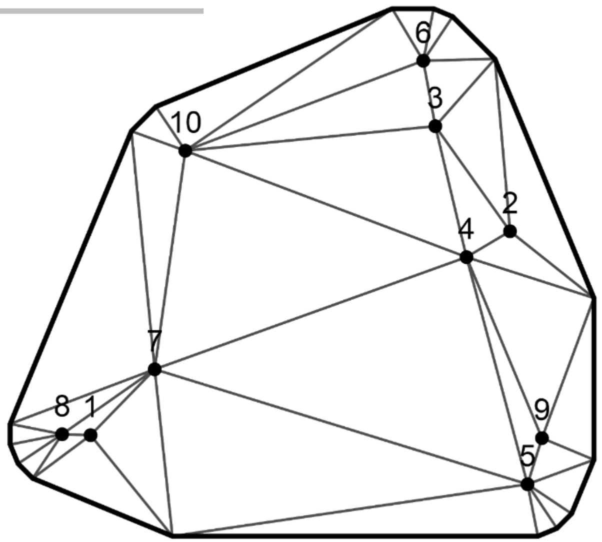


Image source: [Krainski et al. 2019]

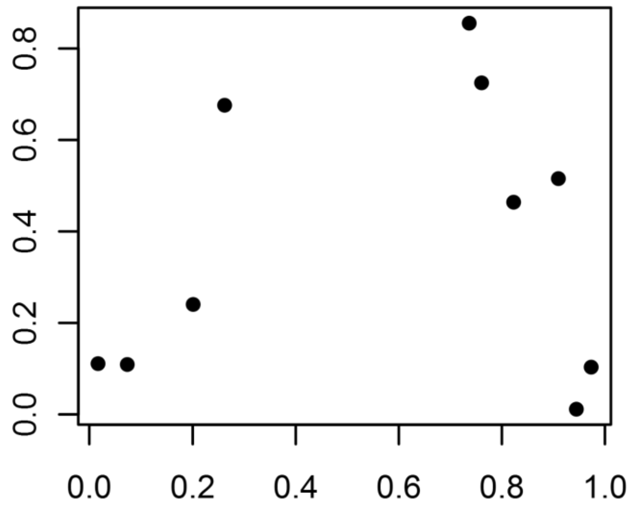
FEM-SPDE with R-INLA

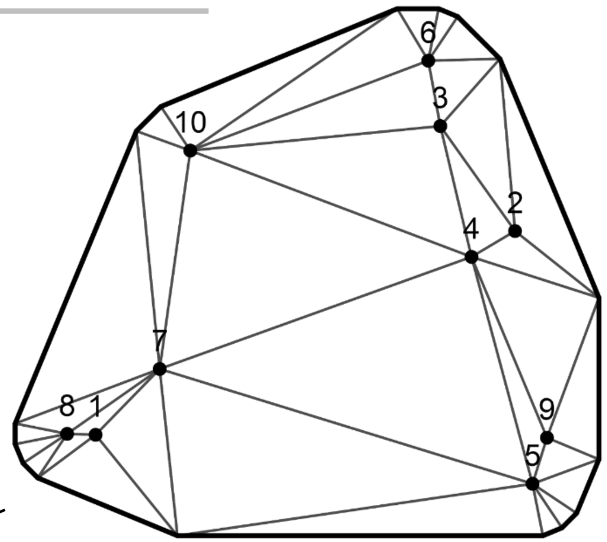



```
inla.mesh.2d()
inla.mesh.create()
```

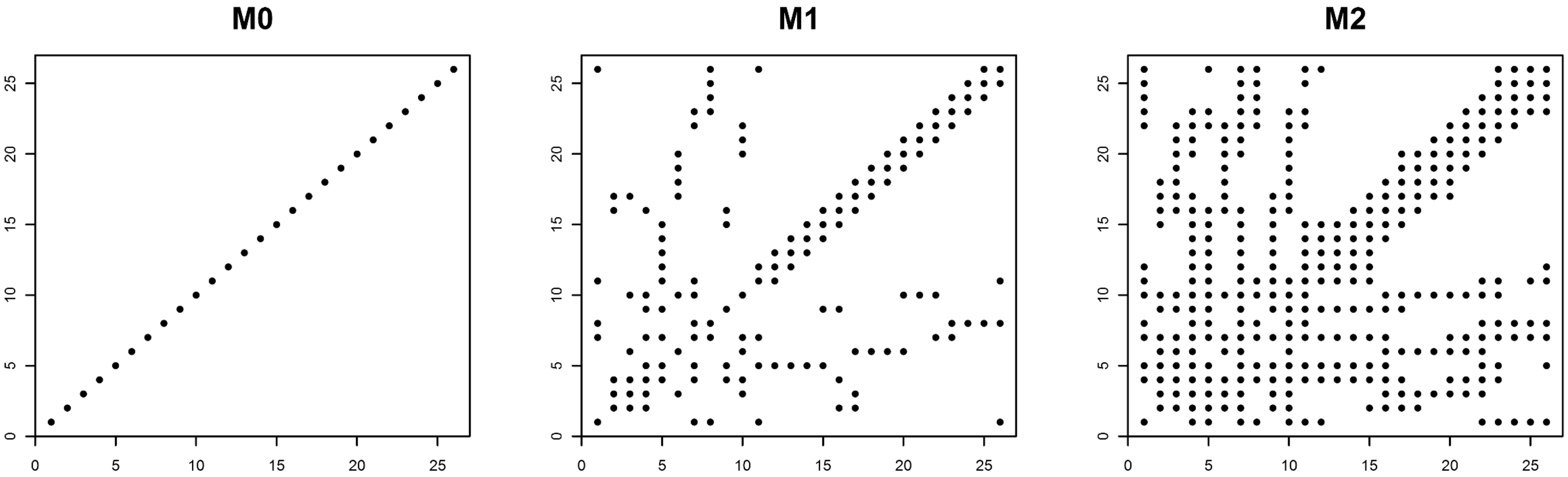


FEM-SPDE with R-INLA

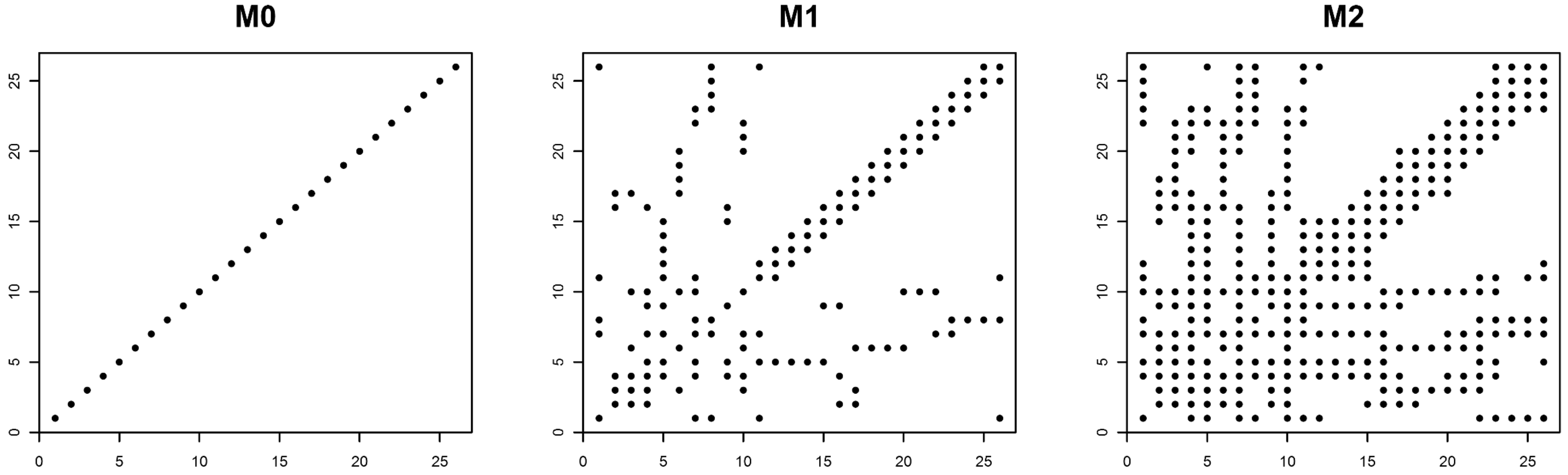




`inla.spde2.matern()`



Precision matrix from R-INLA output



$$Q = \tau^2(\kappa^4 M_1 + 2\kappa^2 M_1 + M_2)$$

$$\omega \sim GMRF(Q)$$

κ: rate of decay in spatial correlation

$$\tau^2 = \frac{1}{4\pi\kappa^2\sigma^2}, \nu = 1$$

TMB's Spatial Functionality

- Covariance Matrix, Σ:
  - matern()
  - MVNORM()
  - AR1()
- Precision Matrix, Q:
  - R_inla::Q_spde()
  - GMRF()
- Sparsity detection